High-order accuracy, entropy stability and convergence for finite difference methods for hyperbolic conservation laws

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We consider systems of hyperbolic conservation laws in one dimension,

$$u_t + f(u)_x = 0. \tag{1}$$

As solutions of (1) develop discontinuities over time, the equation must be interpreted weakly. To single out the physically correct solution from the (large) set of weak solutions, one enforces the *entropy condition*

$$\eta(u)_t + q(u)_x \leq 0$$

for all entropy pairs (η, q).

Our strategy is to mimic this procedure in a discrete setting, with the aim of designing high-order accurate finite difference methods

$$\frac{d}{dt}u_i + \frac{1}{\Delta x}\left(F_{i+1/2} - F_{i-1/2}\right) = 0$$

that converge to the entropy solution. Specifically, we design *entropy stable* methods – finite difference methods that satisfy a discrete entropy inequality

$$\frac{d}{dt}\eta(u_i) + \frac{1}{\Delta x}\left(Q_{i+1/2} - Q_{i-1/2}\right) \leq 0$$

for any given entropy pair (η, q). Combining high-order accurate entropy conservative methods with an ENO (Essentially Non-Oscillatory) reconstruction in entropy variables, we obtain high-order accurate, computationally effective, parameter-free finite difference methods.

To conclude that the method converges strongly one needs a bound on the spacial variation of the solution, in addition to an L^∞ bound. This comes in the form of a weak TV (total variation) bound. The entropy stability of our method implies a seemingly weaker bound, formulated in terms of reconstructed values. We show that for certain orders of reconstruction, this bound is enough to obtain weak TV bounds. Through a compensated compactness argument we conclude that our method converges strongly to a weak solution of (1).

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