A high-order non-linear multiresolution scheme for stochastic-PDE

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Abstract: In the context of the solution of stochastic partial differential equations, a numerical scheme based on a multiresolution representation of data in the stochastic space and an ENO reconstruction operator, is presented. The key feature of the proposed scheme is a strategy inspired by the classical Multiresolution framework proposed by Harten [2], and extended to the stochastic space. The combination of this strategy with a fixed discretization in the physical space, leads to an automatic refinement/derefinement algorithm in the combined physical/stochastic space. The deterministic scheme is a MUSCL-Hancock second-order in space and time, while a third-order reconstruction is used in the stochastic space. This method is applied to several problems of increasing complexity, *i.e.* from the linear advection equation to the 1D non-linear elasticity problem both in homogeneous and heterogeneous media. Results are compared with some reference non-intrusive techniques, like Monte Carlo and Polynomial Chaos Methods.

Keywords: Multiresolution, ENO reconstruction, Finite Volume, Stochastic Partial Differential equations

1 Introduction

In the last decades, a strong effort has been devoted to develop accurate and robust numerical schemes for differential equations. Moreover, differential equations with random coefficients or initial and boundary conditions, play an important part in engineering and physics. The effect of the variability of such uncertain parameters can affect the numerical simulation with a preeminent role in high non-linear problems, such as for example in Computational Fluid Dynamics.

There are several approaches permitting to propagate randomness in numerical simulations. One of the most used is the classical Monte Carlo method. Recently, a Polynomial Chaos (PC) approach has been proposed (see [1, 3]), where the solution is expressed as a truncated polynomial expansion in the spectral space. Its accuracy is dependent on the order of the polynomials reconstruction and of the truncation. However, whenever these techniques are emerging as well-established method to solve stochastic pde, their employment should be limited to problems with a smooth solution. In particular, due to its spectral representation, PC could not converge or have a slower convergence ratio in presence of discontinuous solution.

Note that the presence of discontinuous solution in the physical space could induce also a discontinuous response in the stochastic space. This means that solving accurately the discontinuous surface in the combined space could lead to an infeasible number of calculations, even for 1D spaces. For this reason, we developed a multiresolution inspired techniques based on the compressed representation of data in the stochastic space allowing to retain the accuracy prescribed on the finest resolution level with only a fraction of the total degree of freedom. In particular, in order to achieve a higher reconstruction in the stochastic space, a cubic interpolator is introduced employing an ENO-based stencil to ensure the best reconstruction even in presence of discontinuous solutions.

Due to its intrinsically intrusive behavior, the proposed scheme should be implemented directly in the deterministic code. Anyway, only very light modifications are needed, while preserving the number of equations (not such as in PC intrusive methods).

The deterministic scheme is formulated as a second-order in time and space TVD finite volume approach, namely the MUSCL-Hancock [4] scheme, with different limiting procedure (slope limiters and limited slopes). To show the efficiency and the flexibility of the overall scheme different unsteady problems will be presented. In particular, the basic algorithm will be presented for the scalar advection problem and for the Burgers equation. The validity of the approach will also be shown for vectorial cases solving the Lagrangian elastic problem in a rod with both homogeneous and heterogeneous material. Both the case of linear and non-linear constitutive relations for the material will be addressed.

References

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