High order Stochastic Finite Volume and Stochastic Discontinuous Galerkin methods for the uncertainty quantification in multidimensional conservation laws with random fluxes and initial data

S. Tokareva[†*], **S.Mishra**[†], **C. Schwab**[†]

[†]Seminar for Applied Mathematics, ETH Zurich, Switzerland
Raemistrasse 101, 8092 Zurich
svetlana.tokareva@sam.math.ethz.ch

Keywords: Stochastic Finite Volume method, Stochastic Discontinuous Galerkin method, uncertainty quantification, stochastic conservation laws, computational fluid dynamics.

ABSTRACT

We present and analyze the high order Stochastic Finite Volume (SFV) and Stochastic Discontinuous Galerkin (SDG) methods used to quantify the uncertainty in hyperbolic conservation laws with random initial data and flux coefficients. The direct application of these methods is the simulation of gas flows with uncertain physical parameters and/or initial flow conditions. Many efficient numerical methods have been developed to approximate the entropy solutions of systems of conservation laws, however, in many practical applications it is not always possible to obtain exact physical data due to, for example, measurement or modeling errors. We describe incomplete information in the conservation law mathematically as random fields. Such data are described in terms of statistical quantities of interest like the mean, variance, higher statistical moments; in some cases the distribution law of the stochastic initial data is also assumed to be known. There exist several techniques to quantify the uncertainty (i.e. determine the mean flow and its statistical moments), such as the Monte-Carlo (MC), the Multi-Level Monte Carlo (MLMC) and Stochastic Galerkin method. We present a novel approach to the uncertainty quantification in the conservation laws, the Stochastic Finite Volume Method (SFVM), which is based on the finite volume framework. The SFVM is formulated to solve numerically the system of conservation laws with sources of randomness in both flux coefficients and initial data. Next, we formulate the Stochastic Discontinuous Galerkin method which we primarily use to solve the multidimensional stochastic conservation laws on unstructured grids. Finally, we compare the efficiency of the SFV and SDG methods with of Monte-Carlo type methods.

Consider the hyperbolic system of conservation laws with random flux coefficients and initial data:

$$\partial_t \mathbf{U} + \nabla_x \cdot \mathbf{F}(\mathbf{U}, \omega) = \mathbf{0},\ t > 0; \tag{1}$$

$\mathbf{x} = (x_1, x_2, x_3) \in D_x \subset \mathbb{R}^3$, $\mathbf{U} = [u_1, \ldots, u_p]^T$, $\mathbf{F} = [\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3]$, $\mathbf{F}_k = [f_1, \ldots, f_p]^T,\ k = 1, 2, 3,$ and random initial data $\mathbf{U}(\mathbf{x}, 0, \omega) = \mathbf{U}_0(\mathbf{x}, \omega),\ \omega \in \Omega$.

To construct the high order SFV method, we parametrize the equations by means of the random variable $\mathbf{y} = \mathbf{Y}(\omega) \in \mathbb{R}^q$ and apply the finite-volume discretization in both physical and parametrized stochastic variables. The method then results in the solution of the following ODE system with respect to the cell averages:

$$\frac{d\bar{\mathbf{U}}_{ij}}{dt} + \frac{1}{|K_x^i|}\bar{\mathbf{F}}_{ij}(t) = \mathbf{0}, \tag{2}$$

with $|K_x^i|$ being the physical cell volume, for all $i = 1, \ldots, N_x,\ j = 1, \ldots, N_y$, where N_x and N_y denote the number of mesh elements in the physical and stochastic space, respetively. The flux $\bar{\mathbf{F}}_{ij}$ is obtained by the integration over the cell in the stochastic space. Therefore, the high-order scheme is provided by the high order solution reconstruction (e.g. ENO/WENO) in both physical and stochastic variables combined with the suitable quadrature rule.

We apply the SFVM to solve the Sod's shock tube problem for the Euler equations with randomly distributed initial discontinuity position, random right state and random flux. The results are illustrated in Fig. 1, in which the solution mean (solid line) as well as mean plus/minus standard deviation (dashed lines) are presented.

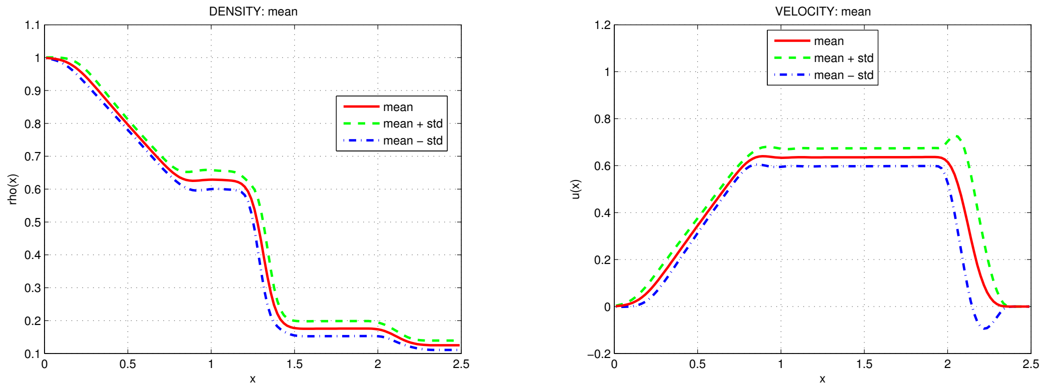

Figure 1: Stochastic Sod's shock tube problem: density (left) and velocity (right)

We next generalize our approach by applying the DG approximation in the physical space and combining it with the FV discretization in the stochastic space. We apply both SFV and SDG methods to study the stochastic flow in the forward-facing step channel. In this simulation, the Mach number of the incoming gas flow is a uniformly distributed random variable, $\mathrm{M} \sim \mathcal{U}[2.95, 3.05]$. The typical deterministic solution for the density as well as the mean and variance of the corresponding random solution are illustrated in Fig. 2.

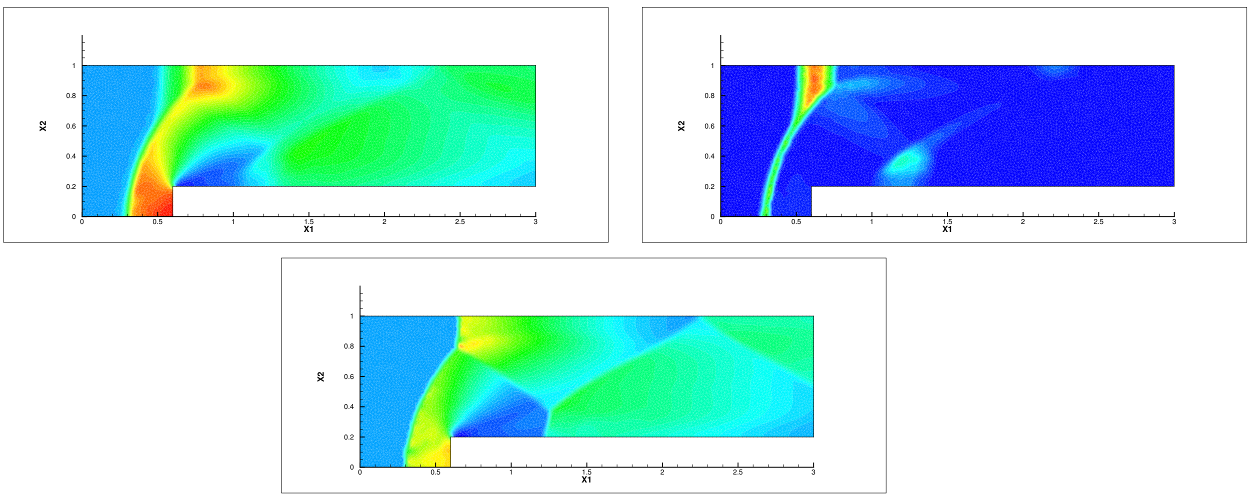

Figure 2: Flow in the forward-facing step channel: density mean (top left), variance (top right) and deterministic solution (bottom)