Taylor expansion method for linear lattice Boltzmann schemes with an external drift. Application to boundary conditions

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Our communication will be divided into two parts. In the first part, we show that it is possible to get the macroscopic fluid equations of lattice Boltzmann schemes with an external force using Taylor expansion. In a second part of our contribution, we validate this general expansion by a detailed application to boundary conditions.

A lattice Boltzmann scheme is defined through the evolution of a population $\{f_i\}$ of q discrete velocities where time, space momentum are discretized. The population evolves in a succession of collision and propagation steps on the nodes of a regular lattice in d dimensions, parametrized by a spatial step Δx. The time step Δt is determined thanks to the acoustic scale λ (*i.e.* $\Delta t = \frac{\Delta x}{\lambda}$). For the D$dQq$ scheme, we note $(v_j)_{0 \leq j \leq q-1}$ the set of q velocities and we assume that for each node x and each velocity v_j, the vertex $x - v_j \Delta t$ is also a node of the lattice. So a time step of a lattice Boltzmann scheme can be written as :

$$f_i(x, t+\Delta t) = f_i^*(x - v_i \Delta t, t), 0 \leq i \leq q-1 \,, \tag{1}$$

where f_i^* is the velocity population after collision. As proposed by d'Humières [3], we introduce the moments m defined by $m = M {\bullet} f$ where M is a given invertible matrix. So in the moment space it is easier to describe the collision step. The moment vector is composed of two kinds of quantities : the first one of conserved variables $V \in \mathbb{R}^N$ which are not affected by the collision step when there is no forcing term. The second one of non conserved quantities Y relax in the collision step. So the moment vector becomes

$$m = \begin{pmatrix} V \\ Y \end{pmatrix}.$$

The lattice Boltzmann schemes with force term are replaced by the following steps :

- The conserved variable are given by : $W = V + \theta \, \Delta t \, F$, where θ is a fixed scalar in $]0, 1[$ and F is a given drift term.
- The relaxation step is performed in the moment space as follows : $Y^* = (\mathrm{Id} - S) \, Y + S \, Y^{eq}$, where S is the diagonal matrix of the relaxation times $s_k, N+1 \leq s_k \leq q-1$ with $0 < s_k < 2$.

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The equilibrium distribution is given by $Y^{eq} = E.W$, where E is a fixed matrix with $q-N$ lines and N columns.

- Due to the force term, the conserved variables during the collision step, evolve according to : $V^* = V + \Delta t\, F$.

We rewrite the sheme (1) in moment space and we obtain the following equation :

$$(2) \qquad \begin{pmatrix} V \\ Y \end{pmatrix} (t+\Delta t) = \sum_{n\geq 0} (\Delta t)^n \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \begin{pmatrix} V \\ Y \end{pmatrix} (t) + \sum_{n\geq 0} (\Delta t)^{n+1} \begin{pmatrix} G_n \\ H_n \end{pmatrix} F.$$

We extend the "Berliner version" [1] of the Taylor expansion method for the expansion (2) when an external forcing term is present. We suppose here that the conserved variables W satisfy a partial differential equation :

$$(3) \qquad \partial_t W = \alpha_1 W + \Delta t\ \alpha_2 W + (\Delta t)^2\ \alpha_3 W + \cdots + \gamma_0 F + (\Delta t)\gamma_1 F + \dots$$

and the non conserved moments Y follow a dynamics of the type

$$(4) \qquad Y = EW + \Delta t\ \beta_1 W + (\Delta t)^2\ \beta_2 W + \cdots + \Delta t \rho_0 F + (\Delta t)^2 \rho_1 F + \dots$$

We entirely specify the development up to order two on Δt. By doing Taylor expansion and by identification with the hypothesis (3), we obtain

$$\begin{cases} \alpha_1 = A_1 + B_1 E, \quad \alpha_2 = B_1\beta_1 + A_2 + B_2 E - \frac{\alpha_1^2}{2}\,, \\ \gamma_0 = G_0 \text{ and } \gamma_1 = B_1\rho_0 + G_1 - \theta A_1 - \frac{1}{2}\left(\alpha_1\gamma_0 + (\gamma_0 - 2\theta)\partial_t\right)\,. \end{cases}$$

For the equation of the N non-conserved moments we have :

$$\begin{cases} \beta_1 = S^{-1}\left[C_1 + D_1 E - E\alpha_1\right] \\ \beta_2 = S^{-1}\left[D_1\beta_1 - E\alpha_2 - \beta_1\alpha_1 - E\frac{\alpha_1^2}{2} + C_2 + D_2 E\right], \\ \rho_0 = S^{-1}\left[H_0 - E\gamma_0 - \theta S E\right] \\ \rho_1 = S^{-1}\left[D_1\rho_0 + H_1 - E\gamma_1 - \beta_1\gamma_0 - \frac{1}{2}E\alpha_1\gamma_0 - \frac{1}{2}E\gamma_0\partial_t - \rho_0\partial_t - \theta C_1\right]\,. \end{cases}$$

Using the Taylor expansion method with an external force given by the above method, we analyze in this contribution a family of situations (1D, 2D) of diffusion and linear fluid problems with "bounce back " and/or "anti-bounce back" numerical boundary conditions. The result is that "magic" parameters proposed in [4] and explored as "quartic" in [2] depend not only on the detailed choice of the moments but also on the parameter θ (which is related to how the drift term is applied). We give a new result for quartic condition for Poiseuille flow with an external drift for the D2Q13 scheme : $40\sigma_5\sigma_9 - 7 - 8\sigma_5 + 16\sigma_5\theta = 0$, where $\sigma_i = \left(\frac{1}{s_i} - \frac{1}{2}\right)$ is the Hénon's parameter.

References

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