A new class of finite volume methods;
an abstract submitted to HONOM 2013

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We introduce a new class of finite volume method, that we refer to as "Active Flux" methods. These differ from regular finite voume methods because the interface fluxes become independent degrees of freedom. Because these d.o.f. are shared between cells, storage requirements are low. The methods will be described for linear advection or linear acoustics in any number of dimensions, and have the following characteristics.

1. Conservative,
2. Implemented on unstructured simplicial grids,
3. Fully explicit, single-step time-marching,
4. Stability up to the maximum time step compatible with physics,
5. Based on the multidimensional physics expressed by spherical means,
6. No one-dimensional ingredients,
7. Very compact stencil, with reconstruction inside a cell independent of all other cells,
8. Notable insensitivity to mesh quality.

The lowest-order version is third-order, and results will be shown for this case, although extension to arbitrary order is possible, at least for linear problems.

For one-dimensional linear advection the method reduces to Scheme V in van Leer [1]. The degrees of freedom then are the average value in each cell, and the point value at each interface, allowing a quadratic reconstruction within cells. From this reconstruction we can deduce the total flux through the downwind interface during the time step, and the new interface value. Then a conservation integral over the cell yields the new cell average. The overall accuracy of the scheme is third order. Dissipation and dispersion plots for this scheme are shown in Figure 1, comparing it with a regular third-order finite-volume scheme.

For higher-dimensional problems, in addition to the cell average, we use point values at each vertex and at the midpoint of each edge, as in Fig 2. These allow for the reconstruction of a quadratic function, together with a "bubble function" that ensures conservation. For advection problems with a given flow direction, we simply trace back the characteristic from each edge location into the appropriate cell, and interpolate to its origin. After updating all edge values like this, we use numerical integration to establish the new cell average. This average determines the new ampltude of the bubble function, and ensures conservation.

For problems that obey a wave equation, $\mathbf{u}_{tt} = a^2\nabla^2\mathbf{u}$ or one of its close relatives, the procedure makes use of the result [2] that the solution $\mathbf{u}(\mathbf{x}, t)$ depends only on data at $t = 0$ lying within a distance at from the point $\mathbf{x}$. In three dimensions, the only relevant data lies on the surface of the sphere with that radius, and in two dimensions in the disc with that radius. There are integals of this data ("spherical means") that give the exact solution. Our method in two dimensions consists of drawing such a disc around each edge or vertex point, as in Figure 3, and computing the integral. This disc will have segments in various cells, and the reconstruction within each of those cells will be used for the contribution made to the integral by that segment.[1] For one-dimensional problems on one-dimensional grids we recover the method of characteristics. We stress however that we are not solving "multidimensional Riemann problems"; our reconstructions do not contain discontinuities. A similar technique apples to various related equations, such as the heat equation, the telegraph equation, Maxwell's equations, and the equations of linear elasticity.[2]

Since the final step in the scheme is a straightforward flux integration, sophisticated analysis such as this enters only into the prediction of the fluxes at edges and vertices. These predictions do not have to be conservative, and need only be accurate to second-order. There is therefore great flexibility in how they are done, and in how nonlinear limiting is applied to them.

In Figure 4, we show the spreading of an initial pressure pulse on a rather coarse unstructured grid. The symmetry of the solution is maintained extremely well. Third-order accuracy is verified in Figure 5.

By the time of the meeting, we hope to have extended this method to the nonlinear Euler equations.

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[1]At present we have computed the integrals analytically, but intend to develop Gauss-type numerical formulas.

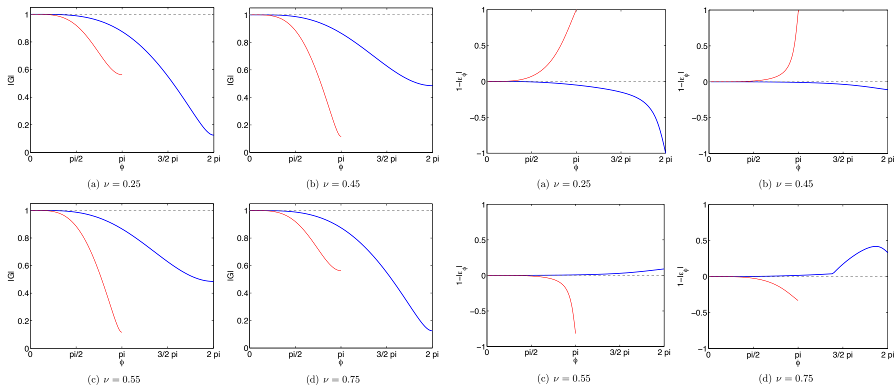

Figure 1: Dissipation(left) and dispersion (right) errors for linear advection. Within each block, the Courant numbers are 0.25, 0.45, 0.55, 0.75. The maximum frequency is 2π instead of π because the storage locations are doubled.

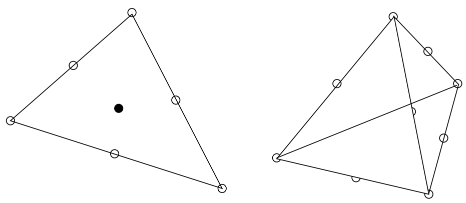

Figure 2: Degrees of freedom in two and three dimensions

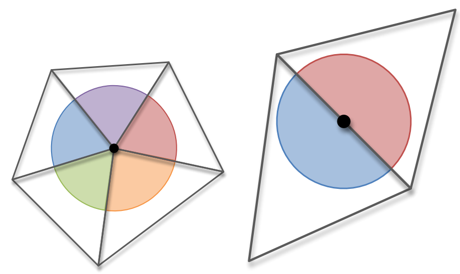

Figure 3: In two dimensions the domain of dependence is a disc centered where the update is required.

References

[1] van Leer, B., Toward the ultimate conservative differencing schemeI V; a new approach to linear advection, *J. Comp Phys*, **23**, pp276-299, 1977.

[2] Courant, R., Hilbert, D., *Methods of mathematical physics, vol II*, Wiley, 1953.

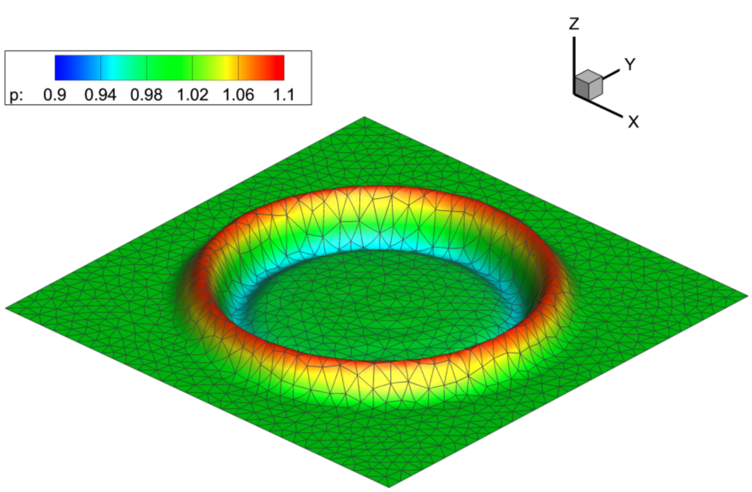

Figure 4: Outcome of an initial Gaussian pressure pulse, calculated on a coarse, fully unstructured grid.

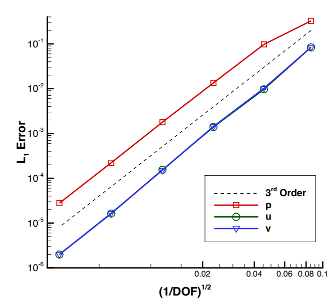

Figure 5: Demonstrating third-order accuracy on an acoustic problem.