

Respect of Gauss' law in Maxwell-like equations with a Nonconforming Discontinuous Galerkin Method

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Abstract

When solving the system of Maxwell's equations, the divergence constraint equations are often neglected. In a simple case, such as when modelling modes, this does not cause an apparent problem. But when a right hand side is added, such as a current term source in the simulations of Maxwell-Vlasov, non physical phenomena related to charge conservation can appear.

We present correction methods that take into account Gauss' law. The novelty of our approach is that we apply these corrections on meshes having an area of sub-mesh. Numerical simulations, using a Discontinuous Galerkin space discretization method, will be performed in a pure Maxwell case and in the future for the coupling of Vlasov-Maxwell equations with a Particle-In-Cell (PIC) method.

Introduction

The Maxwell system (1) allows us to determine the evolution of the electromagnetic fields \mathbf{E} and \mathbf{B} from both data charge and current density, ρ and \mathbf{J} ,

$$\partial_t \mathbf{E} - c^2 \mathbf{rot} \mathbf{B} = -\frac{\mathbf{J}}{\epsilon}, \quad (1a)$$

$$\partial_t \mathbf{B} + \mathbf{rot} \mathbf{E} = 0, \quad (1b)$$

$$\mathbf{div} \mathbf{E} = \frac{\rho}{\epsilon}, \quad (1c)$$

$$\mathbf{div} \mathbf{B} = 0, \quad (1d)$$

where ϵ is the electric permittivity and c is the speed of light. We call equations (1c) and (1d) the divergence constraint equations.

For all times $t \geq 0$, when ρ and \mathbf{J} satisfy the charge conservation equation

$$\partial_t \rho + \mathbf{div} \mathbf{J} = 0. \quad (2)$$

and when the initial \mathbf{E} and \mathbf{B} satisfy (1c) and (1d) the Maxwell system (1a)-(1b) has a unique solution.

In the system of Maxwell's equations (1), data \mathbf{J} can have various origine.

- In a pure Maxwell case, sources (\mathbf{J}, ρ) is analytically given. This is why in practice if you consider data which respect (2) in discrete sense, most numerical methods only solve the equations (1a)-(1b).
- The coupling of the Vlasov-Maxwell equations allows to model the movement of charged particles in an electromagnetic field. Moving particles induce a current term source \mathbf{J} to be introduced into the Maxwell equations (1) to determine fields \mathbf{E} and \mathbf{B} seen by the particles. These fields are using to determine the new positions \mathbf{x} and velocities \mathbf{v} of particles with the Vlasov equation (3).

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_{\mathbf{v}} f = 0, \quad (3)$$

where the distribution function f described the density of particles in the phase space (\mathbf{x}, \mathbf{v}) .

The current term source \mathbf{J} is determined by a deposition method. The deposition leads to errors in charge conservation equation (2) which can become very large in the course of time.

In this article, we present correction methods for the Maxwell system with current source term that respects the Gauss' law. These methods are based on a generalized Lagrange multiplier [1]. The novelty is that we implement and validate these methods on cartesian grids with an area of sub-mesh. The nonconforming mesh allows us to have hanging nodes, i.e., a mesh can have more than one neighbor in one space dimension. An example of a nonconforming mesh is shown on figure 1. We choose these meshes to model multi-scale problems and problems of space charge which are observed in the emission region of a particle beam. In view of the mesh specificity, the numerical method that we have chosen is the Discontinuous Galerkin time domain (DGTD) method.

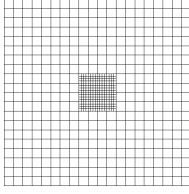


Figure 1: nonconforming cartesian mesh

1 Correction techniques

To satisfy charge conservation, correction techniques consist in adding divergence constraint of Gauss' law to the evolution equation for electrical field \mathbf{E} by a generalized Lagrange multiplier Φ . We solve the following Maxwell system,

$$\partial_t \mathbf{E} - c^2 \text{rot} \mathbf{B} + c^2 \text{grad} \Phi = -\frac{\mathbf{J}}{\epsilon}, \quad (4a)$$

$$\partial_t \mathbf{B} + \text{rot} \mathbf{E} = 0, \quad (4b)$$

$$g(\Phi) + \text{div} \mathbf{E} = \frac{\rho}{\epsilon}, \quad (4c)$$

The new variable Φ defines an additional degree of freedom which respects the following equation

$$\partial_t g(\Phi) - c^2 \Delta \Phi = \frac{1}{\epsilon} (\partial_t \rho + \text{div} \mathbf{J}). \quad (5)$$

We use two choices for the differential operator $g(\Phi)$.

- *Hyperbolic-elliptic formulation* : This formulation corresponds to a Boris' correction [2]. In this approach, we consider in the reformulated Maxwell equations that

$$g(\Phi) = 0. \quad (6)$$

This implies, by the equation (4c), that we need to check exactly the Gauss' law. That is why this correction is considered such as the reference correction. But the elliptic correction is costly in computation time because it requires the inversion of a Laplace operator for each time step.

- *Purely hyperbolic approach* : this approach was introduced by Munz et al. [3]. We choose

$$g(\Phi) = \frac{1}{\chi^2} \partial_t \Phi. \quad (7)$$

This correction is less expensive but less accurate than the elliptic correction method.

The choice of a non conforming DGTD solver for the Maxwell's equations involved a special adaptation for corrections. It is this adaptation that we want to present at the conference in details.

2 Numerical results

Stock et al. in [3] have proven the effectiveness of the hyperbolic correction on a cartesian conforming mesh. We take the Maxwell test case of their paper to apply on the nonconforming mesh illustrate in Figure 1. Figure 2 shows that a correction is needed to keep Gauss' law. Corrections preserve their effectiveness on nonconforming mesh.

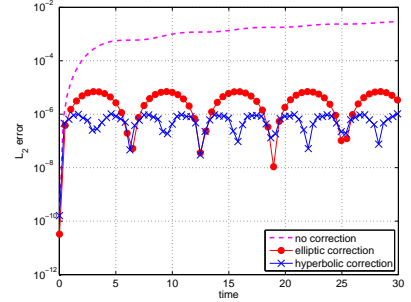


Figure 2: L_2 error on the Gauss' law on nonconforming mesh

Conclusion

It appears necessary to take into account the divergence constraint of Maxwell's system. Our study shows that the corrections are also robust on conforming or nonconforming mesh. In future work we want adapt the correction methods to non conforming mesh for the simulation of plasma with Vlasov-Maxwell equations for which the respect of charge conservation is crucial.

References

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