DIFFUSION-UNIFORM ERROR ESTIMATES FOR NONLINEAR SINGULARLY PERTURBED PROBLEMS IN FINITE ELEMENT METHODS.

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This work is concerned with the analysis of the higher order conforming finite element method (FEM) and discontinuous Galerkin (DG) method applied to the nonstationary singularly perturbed convection-diffusion problem defined in $\Omega \subset \mathbb{R}^d$ with mixed boundary conditions:

$$
\begin{aligned}
&\text{a)} \quad \frac{\partial u}{\partial t} + \operatorname{div} \mathbf{f}(u) = \varepsilon \Delta u + g \quad \text{in } \Omega \times (0,T),\\
&\text{b)} \quad u\big|_{\Gamma_D \times (0,T)} = u_D,\\
&\text{c)} \quad \varepsilon \frac{\partial u}{\partial n}\big|_{\Gamma_N \times (0,T)} = g_N,\\
&\text{d)} \quad u(x,0) = u^0(x), \quad x \in \Omega.
\end{aligned}
$$

Our aim is to derive apriori error estimates in the $L^\infty(L^2)$-norm which are uniform with respect to $\varepsilon \to 0$ and are valid even for the limiting case $\varepsilon = 0$. In the case of linear advection-diffusion this has been done e.g. in [2]. In the nonlinear case, for various explicit time discretizations of the DG scheme, such an error analysis was presented in a series of papers starting with [4]. The typical result for a k-th order explicit scheme is of the form:

Lemma 1. *Let $\mathbf{f} \in [C^2(\mathbb{R})]^d$ and the polynomial order used is $p > 1 + d/2$. Then for sufficiently small h and τ satisfying some CFL-like condition, the error e_h^n of the DG scheme at time level t_n satisfies*

$$\|e_h^n\|_{L^2(\Omega)} \le C(h^{p+1/2} + \tau^k), \quad n = 0, \cdots, N, \tag{1}$$

where $C > 0$ is independent of ε, h, τ.

The proof relies on a nonstandard estimate of the convective terms derived in [4] for the 1D case for periodic or compactly supported solutions under the assumption that the numerical flux is an *E-flux*. Using this estimate, if we know *apriori* that $\|e_h^n\| = O(h^{1+d/2}), n = 0, \cdots, N$, then we may prove the improved estimate (1). A bootstrapping argument using mathematical induction is then applied in order to eliminate the *apriori* assumption.

Since the proof relies heavily on mathematical induction, the technique cannot be directly applied to estimates for the method of lines (no discrete structure with respect to time) and implicit discretizations (not enough *apriori* information about the solution on the next time level).

In the presented work, we use generalized versions of the analysis of the convective terms derived in [4] and similar estimates for the FEM to prove corresponding versions of Lemma 1 in the case of space semidiscretization and a backward Euler implicit scheme. For the DG method, this has been done by the author in [3].

Method of lines. Here we apply two different techniques. First, we use the so-called *continuous mathematical induction* [1] instead of standard mathematical induction in the bootstrapping argument. This is a technique that we shall also use in the implicit case. Alternatively, we prove the same result using a nonlinear variant of Gronwall's inequality. It can be proven that the latter technique has no discrete counterpart in the case of implicit time discretization.

Implicit time discretization. First, we prove that for the implicit scheme an estimate similar to Lemma 1 cannot be proved only from the error equation and the considered estimates of its individual terms. Hence, we need to supply more information about the properties of the problem and its (discrete) solution in order to derive the desired error estimates.

In the case of space semidiscretization we used the continuity of the error with respect to time which allows us to apply either continuous mathematical induction or a suitable Gronwall lemma. In the implicit case, we introduce a *continuation* $\widetilde{e}_h : [0,T] \to L^2(\Omega)$ of the error $e_h^n, n = 0, \cdots, N$ constructed by means of a suitable modification of the discrete problem. We prove that estimates for this continuated solution directly imply estimates for the original implicit solution. The fact that $\widetilde{e}_h$ is continuous with respect to time and that it relates to the structure of the problem allows us to prove estimates for $\widetilde{e}_h$ via continuous mathematical induction. These estimates directly give us the desired estimates for e_h^n.

To conclude, the presented technique represents a new and simple tool for the analysis of nonlinear convective and singularly perturbed problems. The technique using continuous mathematical induction allows for the rigorous treatment of *locally* Lipschitz continuous convective nonlinearities, a point which is not treated quite rigorously in [4]. A principal artefact of using this technique is that we obtain a rather restrictive CFL-like condition even in the case of an implicit time discretization and that the results are not valid for the lowest order approximation degrees (we need $p > 1+d/2$). Nonetheless, these results are interesting, since they are valid even in the purely convective case, where there is a lack of results for nonlinear problems. Finally, we note that the FEM version of the error analysis can be quite straightforwardly extended to much more general equations and systems of equations.

References

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