EBR Schemes: New Developments

Ilya Abalakin[1], Pavel Bakhvalov[2], and Tatiana Kozubskaya[1]
[1]*Keldysh Institute of Applied Mathematics RAS, Moscow, Russia*
[2]*Moscow Institute of Physics and Technology, Dolgoprudny, Russia*

The idea of the vertex-centered EBR (Edge-Based Reconstruction) schemes based on quasi-1D reconstruction of variables for unstructured meshes was firstly proposed in the 90ths years of the last century [1], [2] and then has been intensively developed and used in applications [3-5]. Here we would like to raise an interest to these schemes presenting their new interpretation, new properties and new ways of their possible extension.

The main idea of the EBR schemes consists in extracting 1D direction along which the reconstruction of conservative variables (or fluxes) is performed. This direction is defined by the mesh edge, let it be the edge $(i, i+1)$ (Fig. 1).

In the frames of FV approach the vertex-centered schemes need special construction of control volumes surrounding each node of the mesh. The control volumes are used for approximating the conservation laws. So if performing the reconstruction of solution along the line $\mathbf{Edge} = \overrightarrow{i\,i+1}$ (Fig. 1), we can find the approximate values of variables in the point of intersection of the line $\mathbf{Edge}$ with the corresponding face (separating the nodes *i* and *i+1*) of the control. Basing on the values in such points we then calculate the surface integrals using the method of rectangles.

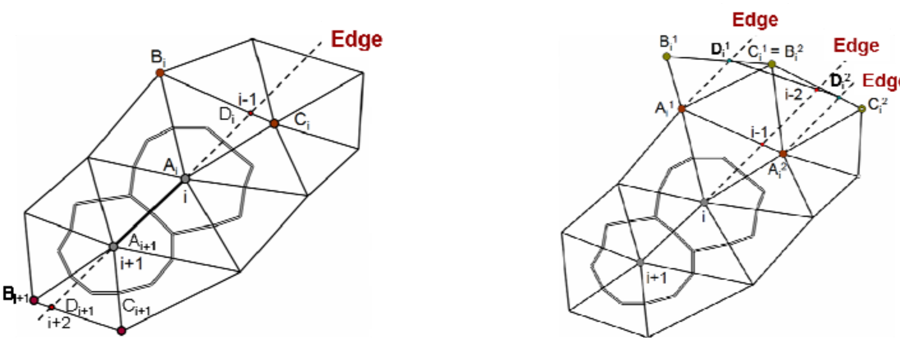

Figure 1: Definition of the direction of reconstruction and stencil points Ω_i for arbitrary triangular mesh.

In order to find the value in the point of intersection of face and straight line $\mathbf{Edge}$, let us reconstruct the variables along this line basing on the nodes *i* and *i+1* and, in the case of high-order reconstruction, a set of additional points also belonging to $\mathbf{Edge}$. The way of defining those additional points, in particular, the points $i-2, i-1, i+2$ is illustrated in Fig. 1. As a result, having the straight line and a set of stencil points $\{\mathbf{r}_k = (x_k, y_k),\ k \in \Omega_i\}$ belonging to it one can find the approximated values of variables in the point of interface. This value can be represented as a linear combination of normalized first finite differences $\Delta_k = (u_k - u_{k-1}) / (\mathbf{Edge} \cdot \mathbf{r}_k - \mathbf{Edge} \cdot \mathbf{r}_{k-1})$ calculated via the values of variables in the stencil points. The coefficients of this combination are found by requiring the coincidence of the approximation with the high-order one in case of 1D finite-difference scheme for uniform mesh.

As it is seen in Fig. 1, the additional stencil points in case of arbitrary triangular (tetrahedral) mesh do not coincide with the mesh nodes. So their values are unknown as well as the finite differences Δ_k based on them. At the same time, the needed finite differences Δ_k can be found with the help of gradients defined on the triangles containing the corresponding segments of the line $\mathbf{Edge}$ (triangles $A_i B_i C_i$, $A_{i+1} B_{i+1} C_{i+1}$) and located along this direction (triangles $A_i^1 B_i^1 C_i^1$, $A_i^2 B_i^2 C_i^2$) as it is shown in Fig 1. More correctly, for this purpose the P1 Galerkin gradients projected to the direction $\mathbf{Edge}$ are taken.

In doing so, we formulate the EBR schemes in terms of first finite differences replacing them with their unstructured analogues. In this sense, the EBR schemes can be also considered as a generalization of FD approach on the case of unstructured meshes.

Let us define the notion of quasi-1D reconstruction more accurately. We say that the reconstruction of variables or fluxes along the chosen direction at the cell interface is quasi-1D if, being applied to the triangular (tetrahedral) mesh consisting of the "equal" elements, it coincides with the genuinely 1D-case reconstruction. Here under the mesh consisting of "equal" elements we mean the meshes invariant under liner translation in the direction of each edge on the distance equal to the length of this edge. Such meshes can be built by applying any affine transformation to the Cartesian triangular or tetrahedral mesh. It can be proved that the EBR schemes on the meshes of the above described family provide the same high order of accuracy as the basic genuinely 1D-case reconstruction. So the class of meshes where the EBR schemes reach their highest theoretical order (up to the sixth) of accuracy is significantly wider than the Cartesian meshes only as we thought before [4].

The idea of quasi-1D reconstruction can be extended to different meshes and geometrical configurations, and offers a lot of attractive possibilities for the further developments.

In particular, we applied the quasi-1D reconstruction to the cell-centered formulation for unstructured meshes. In this case, the basic direction **Edge** connects not the mesh nodes, but the cell gravity centers. The values in the additional points of 1D stencil are also determined with the help of linear interpolation so that the requirement of quasi-1D reconstruction is satisfied.

The EBR scheme gas been also extended to the case of curvilinear construction of mesh lines what is often used in practice, for instance, in boundary layers along the curvilinear surfaces. In this case the high-order quasi-1D reconstruction along the mesh edges can be replaced with the reconstruction along the curves parallel to the surface shape. Such modification improves the accuracy of the numerical result in the regions of strongly curved boundary layers.

The quasi-1D approach underlies the development of many other useful EBR scheme modifications in a way of implementing the techniques well developed for 1D case to the multidimensional case of unstructured meshes. In particular, to support the shock capturing, we elaborated the WENO-EBR scheme [6].

If comparing the EBR schemes both with the classical polynomial-based FV methods and FE schemes used for unstructured meshes, we can say that these schemes are significantly cheaper from the standpoint of computational costs at keeping rather high accuracy of order between second and fifth (sixth in non-dissipative case) depending on the mesh quality.

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