Entropy-stable discontinuous Galerkin finite element method with streamline diffusion and shock-capturing

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ABSTRACT

We consider systems of conservation laws in d spatial dimensions:

$$\mathbf{U}_t + \sum_{k=1}^{d} \mathbf{F}^k(\mathbf{U})_{x_k} = 0 \tag{1}$$

in a spatial domain Ω and t in $[0,T]$ together with suitable boundary conditions. $\mathbf{U}$ is the unknown vector of conserved quantities and $\mathbf{F}^k$ is the flux function in x_k-direction.

The spatial elements are denoted K with $K \in \mathcal{T}$, while $I^n = [t^n, t^{n+1}]$ is the temporal grid with $n \in \{0, \ldots, N-1\}$, $t^0 = 0$ and $t^N = T$.

We work in entropy variables (entropy symmetrisation): Choose an entropy function $S(\mathbf{U})$, then the entropy variables $\mathbf{V}$ are given by $\mathbf{V} = S_{\mathbf{U}}$; so $\mathbf{U}$ is a function of $\mathbf{V}$. The semilinear form for the space-time discontinuous Galerkin finite elements method is

$$\begin{aligned}
\mathcal{B}_{DG}(\mathbf{V},\mathbf{W}) = &-\sum_n \sum_K \int_{I^n} \int_K \left(\langle \mathbf{U}(\mathbf{V}), \mathbf{W}_t \rangle + \sum_{k=1}^{d} \left\langle \mathbf{F}^k(\mathbf{V}), \mathbf{W}_{x_k} \right\rangle \right) dx dt \\
&+ \sum_n \sum_K \int_K \langle \mathbf{U}(\mathbf{V}_{n+1,-}), \mathbf{W}_{n+1,-} \rangle \, dx - \sum_n \sum_K \int_K \langle \mathbf{U}(\mathbf{V}_{n,-}), \mathbf{W}_{n,+} \rangle \, dx \\
&+ \sum_n \sum_K \sum_{K' \in \mathcal{N}(K)} \int_{I^n} \int_{\partial_{KK'}} \langle \mathbb{F}(\mathbf{V}_{K,-}, \mathbf{V}_{K,+}; \nu_{KK'}), \mathbf{W}_{K,-} \rangle \, d\sigma(x) dt
\end{aligned} \tag{2}$$

where $\mathcal{N}(K)$ are the neighbouring cells of cell K, $\partial_{KK'}$ is the common boundary of cell K and K' and $\nu_{KK'}$ is the outward normal of cell K. The numerical flux $\mathbb{F}$ is chosen to be entropy-stable, i.e. it is an entropy-conservative flux [4] together with a numerical diffusion. Using this form in the weak formulation already ensures entropy stability and (formally) arbitrarily high order of accuracy. But as we are interested in solutions with shocks we have to deal with spurious oscillations at discontinuities.

Therefore, we include a streamline-diffusion and a shock-capturing term [2, 3], where the streamline-diffusion term gives some control on the residual while the shock-capturing leads to additional diffusion at shocks. The streamline diffusion term is

$$\mathcal{B}_{SD}(\mathbf{V},\mathbf{W}) = \sum_n \sum_K \int_{I^n} \int_K \left\langle \left(\mathbf{U_V}(\mathbf{V}) \mathbf{W}_t + \sum_{k=1}^{d} \mathbf{F}^k_{\mathbf{V}}(\mathbf{V}) \mathbf{W}_{x_k} \right), \mathbf{D}^{SD} \mathrm{Res} \right\rangle dx dt, \tag{3}$$

where $\mathrm{Res} = \mathbf{U}(\mathbf{V}^{\Delta x})_t + \sum_{k=1}^{d} \mathbf{F}^k(\mathbf{V}^{\Delta x})_{x_k}$ is the intra-element residual and $\mathbf{D}^{SD}$ is a scaling matrix proportional to the mesh width Δx. The shock-capturing term is roughly

$$\mathcal{B}_{SC}(\mathbf{V}, \mathbf{W}) = \sum_n \sum_K \int_{I^n} \int_K D_{n,K}^{SC} \left(\langle \mathbf{W}_t, \mathbf{U}(\mathbf{V})_t \rangle + \sum_{k=1}^{d} \langle \mathbf{W}_{x_k}, \mathbf{U}(\mathbf{V})_{x_k} \rangle \right) dx dt. \tag{4}$$

It adds diffusion proportional to $D_{n,K}^{SC}$ which is an integral quantity of the norm of the residual normalized by the norm of the gradient of $\mathbf{U}$.

Choosing the space of test and trial functions $\mathcal{V}_p$ (piecewise polynomials of degree p) this leads to the weak formulation: Find $\mathbf{V} \in \mathcal{V}_p$ such that

$$\forall \mathbf{W} \in \mathcal{V}_p : \mathcal{B}_{DG}(\mathbf{V}, \mathbf{W}) + \mathcal{B}_{SD}(\mathbf{V}, \mathbf{W}) + \mathcal{B}_{SC}(\mathbf{V}, \mathbf{W}) = 0 \tag{5}$$

Note that because the streamline diffusion and the shock-capturing terms are non-negative for $\mathbf{W} = \mathbf{V}$ entropy stability carries over to this formulation.

We investigate the convergence properties of the method theoretically and experimentally for a range of problems [1]. In particular we have solved the linear advection equation, Burgers' equation, the wave equation and the Euler equations (see e.g. Figure 1) in one or two spatial dimensions.

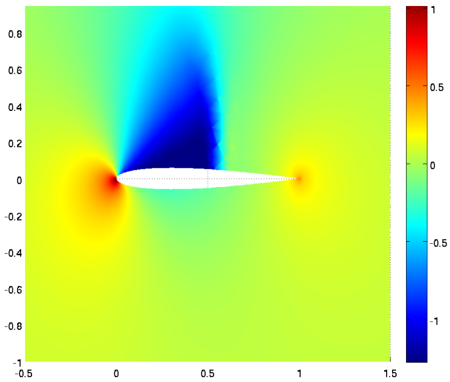

Figure 1: Pressure coefficient C_p for a flow over a NACA 0012 airfoil with $\mathrm{Ma}_\infty = 0.75$, computed using 2809 cells and polynomial degree 2.

An important aspect is how to solve the big nonlinear system for the unknown degrees of freedom efficiently. We investigate different solution methods, including preconditioning of the linearized system of equations.

References

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