Approximate solutions of generalized Riemann problems for nonlinear systems of hyperbolic conservation laws

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Consider the generalized Riemann problem for a hyperbolic conservation law, that is the Cauchy problem

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) = 0, \qquad (x,t) \in \mathbb{R} \times [0,\infty),$$
$$u(x,0) = \begin{cases} \hat{u}_L(x), & \text{if } x < 0, \\ \hat{u}_R(x), & \text{if } x > 0. \end{cases}$$

We assume $\hat{u}_L,\ \hat{u}_R$ to be smooth functions, satisfying $\hat{u}_L(0) \neq \hat{u}_R(0)$.

We are interested in the generalized Riemann problem as a building block for high order finite volume schemes. Following a generalized Godunov approach, the ADER scheme of Toro and Titarev [4] first builds a piecewise smooth reconstruction of the solution at each time-step and then evolves the data by solving generalized Riemann problems.

Toro and Titarev [4] suggested to approximate the solution at the cell-interfaces by a truncated Taylor series expansion in time. The time-derivatives at the origin are then found by first expressing them as functions of spatial derivatives, using a Cauchy-Kowalewskaya procedure. The values of the spatial derivatives are found by solving classical Riemann problems with linearised evolution equations for the spatial derivatives. While this technique has been applied successfully to a broad set of problems, there is few rigorous analysis on the theoretical aspects of this method. Moreover, it was observed by Castro and Toro [1] that the solver of Toro and Titarev can give poor results for problems with large jumps in the initial data.

On the other hand, LeFloch and Raviart [3] have explicitly constructed a local power-series expansion of the solution of the generalized Riemann problem. We have recently shown that for a scalar problem in one spatial dimension with strictly convex flux both the Toro-Titarev solver and the LeFloch-Raviart expansion yield the same truncated Taylor series expansion in time [2].

For nonlinear systems of hyperbolic conservation laws, however, there is a difference. We show analytically that both methods formally construct the same truncated Taylor series expansion. The only difference in both methods is the way spatial derivatives at the origin are found. While the Toro-Titatarev solver uses linearised Riemann problems, in the LeFloch-Raviart expansion the values of those spatial derivatives are obtained through the Rankine-Hugoniot conditions. We show that the difference between the two methods is small when there is only a small jump in the initial data, which explains why the ADER scheme achieves its designed order of accuracy in smooth regions. But the difference may become large when there is a large jump in the initial data, thus explaining the results of Castro and Toro [1].

References

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