

High Order Discontinuous Galerkin Methods for the Simulation of Underresolved Multiscale Problems

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Abstract

High order methods are known for their efficiency when well resolved discretizations are possible. Discontinuous Galerkin methods are a prominent member and allow space and time adaptivity in a natural way [1]. In the context of multi-scale problems, the advantage of a high order accurate discretization is not obvious and even controversial, as due to the range of scales a well resolved discretization is typically not feasible. If the problem is underresolved, conclusions drawn from theoretical analysis with the discretization parameter $h \rightarrow 0$ are not generally applicable. However, convergence order of a discretization is not the only relevant property when approximating a multi-scale problem. The accurate reproduction of individual scales, i.e. the dispersion and dissipation behavior of the spatial derivative operator is more important than the theoretical order of convergence. To emphasize this proposition, we will demonstrate the accuracy of an *analytic* high order discontinuous Galerkin discretization of compressible turbulent flows [3]. We call the method *analytic* in the sense that the evaluation of the inner products is exact.

However, in practical applications there is a desire of computational efficiency, which is typically achieved by approximating the inner products, e.g. with numerical quadrature. A rather efficient variant of such a discrete discontinuous Galerkin scheme is based on collocation type nodal discontinuous Galerkin approximations, e.g. [4]. In those methods, the solution as well as the flux functions are approximated by interpolation polynomials of the same degree, which gives the possibility to construct highly efficient operators. Among the most efficient variants is the so-called discontinuous Galerkin collocation spectral element method (DGSEM), where the flux approximation based on interpolation is collocated with the numerical quadrature used for the evaluation of the inner-products, e.g. [5].

This discretization process has a fundamental impact on the stability of the high order method due to approximation errors of the non-linear terms, typically called aliasing, and its associated instabilities. We will show a novel interpretation of discrete discontinuous Galerkin methods and use this insight to construct stable discrete operators for the non-linear Burgers equation. We formulate a skew-symmetric discontinuous Galerkin method and investigate its energy stability for different common numerical flux functions [2].

As a by-product, the interpretation allows us to combine finite difference, finite element and even spline based operators in a common discontinuous Galerkin type framework. All analysis results carry directly over in this case and allow to

introduce a common (computational) framework for the numerical discretization of hyperbolic partial differential equations.

References

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