High–Order Asymptotic–Preserving Methods for nonlinear hyperbolic to parabolic relaxation problems

Sebastiano Boscarino[†], **Philippe G LeFloch, Giovanni Russo**

[†]Dipartimento di Matematica e Informatica, Università di Catania
Viale A. Doria 6, 95125, Catania, Italy
boscarino@dmi.unict.it

ABSTRACT

We consider nonlinear hyperbolic systems with stiff nonlinear relaxation. In the parabolic scaling, when the stiffness parameter ϵ, the dynamics is asymptotically governed by effective systems which are of parabolic type and may contain possibly degenerate and fully nonlinear diffusion terms.

In [4] we study and analyze several model problems with linear and nonlinear relaxation, including Kawashima-LeFloch's model [5]. The analysis is performed on the simplest nonlinear model problem (see [4] for details)

$$\begin{aligned} u_t + v_x &= 0, \\ \epsilon^2\, v_t + u_x &= -|v|^{m-1}\, v, \end{aligned} \tag{1}$$

where $m > 0$ is a real parameter. As the parameter ϵ vanishes the system reduced to the associated effective equation

$$u_t = \big(|u_x|^{\alpha}(u_x)\big)_x, \quad |v|^{m-1}\, v \quad = -u_x, \tag{2}$$

with $\alpha = -1 + 1/m$. It is natural to distinguish between three different behaviors:

$$\begin{aligned} \text{sub–linear} : 0 < m < 1 \quad & \alpha > 0, \\ \text{linear} : m = 1, \quad & \alpha = 0, \\ \text{super–linear} : m > 1, \quad & \alpha < 0. \end{aligned} \tag{3}$$

The equation (2), although parabolic in nature, may regularize initially discontinuous initial data, and we may expect jump singularities (in the first–order derivative u_x) to remain in the late–time limit. The case $m > 1$ lead to a fast diffusion, which makes explicit scheme in time almost useless, because of the unboundedness of the diffusion coefficient.

In order to solve numerically such problem we use a method of line. First we discretize in space by conservative finite difference methods in the sense of Shu and obtain a large system of ODE's. For such system, we introduce implicit–explicit (IMEX) methods which are based on Runge–Kutta (R–K) discretization in time. We shall show that the use of classical IMEX R-K methods will lead to consistent explicit discretization of the limit equation (2). Because of this, the limit scheme will suffer of the typical parabolic CFL restriction $\Delta t \propto \Delta x^2$. Furthermore in some cases such restriction does not allow the computation of the solution due to the unfoundedness of the diffusion coefficient (super-linear case).

To overcome such drawback one can make use of a penalization technique, based on adding two opposite terms to the first equation in (1), and treating one explicitly and one implicitly, (see for details [4, 1, 2]),

$$\begin{aligned} u_t &= -(v + \mu(\epsilon)|u_x|^{\alpha}u_x)_x + \mu(\epsilon)(|u_x|^{\alpha}u_x)_x, \\ \epsilon^2 v_t &= -u_x - |v|^{m-1}v. \end{aligned} \tag{4}$$

Here, $\mu(\epsilon)$ is such that $\mu : \mathbb{R}^+ \to [0,\ 1]$ and $\mu(0) = 1$. When ϵ is not small there is no reason to add and subtract the term $\mu(\epsilon)u_{xx}$, therefore $\mu(\epsilon)$ will be small in such a regime, i.e. $\mu(\epsilon) \approx 0$. By this technique in the limit $\epsilon \to 0$ the IMEX R-K scheme converges to an implicit method for the first equation in (2), then this IMEX R-K schemes are able to solve the hyperbolic system containing nonlinear relaxation without any restriction on the time step. Two kind of approaches for IMEX R-K scheme will be considered in [4]. The first will be denoted as *partitioned approach* [1] and the corresponding IMEX R-K scheme is called **IMEX-I** R-K scheme. The second approach is denoted *additive* [2] and the corresponding scheme is called **IMEX-E**. In [4] a very efficient IMEX-I R-K scheme has been introduced, in order to solve the first equation in (2) which is a nonlinear diffusion equation. The method we propose is stable, linearly implicit and can be designed up to any order of accuracy.

Several applications to some test problems will be presented in order to show the effectiveness of the new approach.

References

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