DMP-preserving finite element approximation of transport problems based on nonlinear stabilization techniques

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ABSTRACT

Crude Galerkin approximations of hyperbolic problems with continuous finite elements (FEs) are plagued with spurious oscillations. Similar situations happen when considering coercivity loss limits of some elliptic problems, e.g. convection-diffusion-reaction systems. A first remedy is to include additional *linear stabilization* terms to the Galerkin formulation that improve stability while keeping accuracy in smooth regions. The amount of literature on this topic is overwhelming but we can make a distinction between two main types of methods:

- residual-based (VMS) formulations, which include terms that always involve the residual, making the methods consistent,
- projection-based formulations, which include additional stabilization terms that include the difference between some quantities and their FE projections.

The second family of methods have many good properties: we keep symmetry for symmetric problems and no undesirable cross-terms appear for time-dependent problems.

However, linear stabilization techniques are not fully satisfactory, since spurious undershoots and overshoots usually appear locally around shocks or layers with large gradients. The situation is specially dramatic for high order approximations and when nonlinear complex hyperbolic problems are to be solved. Thus, additional *nonlinear stabilization* terms, i.e. shock-capturing (SC) terms, must be included on top of the linear method, in order to improve the situation around shocks and layers. Finite volume and discontinuous Galerkin formulations are usually combined with limiters, which can be understood a a postprocess of the solution; these methods can produce high-quality results, but they can hardly be used as implicit solvers and cannot be naturally extended to continuous FE formulations. On the other hand, continuous FE formulations are usually stabilized by introducing *nonlinear artificial diffusion* terms. We can distinguish among three different types of SC artificial diffusion methods: residual-based viscosity formulations [6], entropy viscosity methods [5] and what we call gradient jump viscosity methods.

In this work, we propose a method with two new ingredients. First, linear stabilization is based on a symmetric projection method based on the Scott-Zhang projector (see [1]); this type of stabilization has recently been proposed to stabilized saddle-point problems. Second, we consider a new SC formulation that has its roots in [2,3]; in fact, switching off the linear stabilization and in the 1D case we recover the method in [2]. This SC technique introduces a viscosity which depends on a discontinuity sensor.

As commented in [4], the introduction of linear stabilization can harm the properties of the SC techniques. We consider a blend of linear and nonlinear terms in such a way that the linear stabilization is

reduced as the nonlinear one is activated. This blend is designed in order to satisfy a discrete maximum principle (DMP) property (at least) for 1D problems. Two alternative formulations are proposed, which differ in the way the blend is designed. Further, one requires a particular mapping from element to node in the Scott-Zhang projector.

On one side, we perform a detailed set of numerical experiments in order to show the following facts:

- The introduction of the linear stabilization certainly improves the method with nonlinear stabilization only
- The resulting method is accurate and robust when compared to residual and entropy-based nonlinear viscosity for linear and nonlinear problems, specially when high order methods are used
- The method can readily be used with discontinuous Galerkin formulations and outperforms nonlinear viscosity methods that have been specially designed for this case
- Implicit formulations are possible, since the nonlinear iterations converge.

The evolutionary linear transport equation, Burger's problem and convection-diffusion reaction problem are considered.

On the other side, we restrict ourselves to 1D problems and first order approximations of (possibly nonlinear) transport problems and prove that the resulting method inherits a (strong) DMP property; these results are also checked experimentally. As far as we know, this is the first FE method which blends linear stabilization with SC and enjoys a DMP.

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